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Stable graphs for a family of endomorphisms☆

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Abstract

We prove several fixed subgraph properties. In particular it is shown that if $\mathfrak{F}$ is a commuting family of contractions of a connected graph G without infinite path and infinite interval, then there exists a nonempty finite subgraph F which is invariant under any element of $\mathfrak{F}$. In particular this subgraph F is a simplex if G is moreover a strongly dismantlable graph or a ball-Helly graph without infinite block, or if it is chordal. This implies that for any commuting family of contractions of a tree without infinite path, there is a common fixed vertex or a common fixed edge.

0. Introduction

In 1973 Halin [3] proved that any endomorphism of a connected rayless graph (i.e., not containing a one-way infinite path) stabilizes a finite set of vertices, thus that any endomorphism of a rayless tree leaves a vertex or an edge invariant; we slighlty improved those results in [9] by replacing endomorphism by contraction, that is a map between the vertices that preserves or contracts the edges. In 1979 Nowakowski and Rival [6] showed that any contraction of a graph G stabilizes a vertex or an edge if and only if G is a rayless tree. Then in [12] Schmidt proved a result, that we improved in [8], stating that any connected rayless graph has a finite subgraph which is invariant under any automorphism. More recently, some similar results dealing with finite invariant subgraphs of special types, such as simplices, hypercubes and Hamming graphs, were obtained by Polat [9, 10], Tardif [13], and Chastand with Polat [1], respectively.

In this paper we continue our investigations of those invariant subgraph properties by considering commuting families of contractions, inspired by two well-known results stating that commuting families of endomorphisms of certain structures have a common fixed point: the Markov–Kakutani Theorem, [5, 4], for compact convex sets of locally convex linear topological spaces, and the Tarski Theorem [14] for complete lattices. We get several results of this kind — Theorem 2.3 (resp.

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Theorems 3.4, 3.7 and 3.10) — which give sufficient conditions for the existence of a common invariant finite subgraph (resp. simplex) for any commuting family of contractions of a connected graph. They imply in particular that, for any commuting family $\mathfrak{F}$ of contractions of a rayless tree, there is a vertex or an edge which is fixed by every element of $\mathfrak{F}$. Very recently, Tardif [13] proved a similar result about common invariant hypercubes in median graphs without distance preserving rays.

1. Preliminaries

1.1. The graphs we consider are undirected, without loops and multiple edges. A complete graph will be simply called a *simplex*. If $x \in V(G)$, the set $V(x;G) := \{y \in V(G)\colon \{x,y\} \in E(G)\}$ is the *neighborhood* of x. For $A \subseteq V(G)$ we denote by $G|A$ the subgraph of G induced by A, and we set $G - A := G|(V(G) - A)$. A *path* $W := \langle x_0, \ldots, x_n \rangle$ is a graph with $V(W) = \{x_0, \ldots, x_n\}$, $x_i \neq x_j$ if $i \neq j$, and $E(W) = \{\{x_i, x_{i+1}\}\colon 0 \leqslant i < n\}$. A *ray* or *one-way infinite path* $R := \langle x_0, x_1 \ldots \rangle$ is defined similarly. A graph is *rayless* if it contains no ray. A path $\langle x_0, \ldots, x_n \rangle$ is called an x_0x_n*-path*, x_0 and x_n are its *endpoints*, while the other vertices are called its *internal* vertices. The set of all xy-paths of G is denoted by $P_G(x,y)$, and we set $G(x,y) := \bigcup P_G(x,y)$. For $A, B \subseteq V(G)$, an AB*-path* of G is an xy-path of G with $x \in A$, $y \in B$ and no internal vertex in $A \cup B$. If A, B and S are subsets of $V(G)$, S *separates* A *from* B *in* G if all AB-paths of G have vertices in S. For $x \in V(G)$ and $A \subseteq V(G)$, and xA*-linkage* of G is a set of xA-paths of G which have pairwise only x in common. If there exists an infinite xA-linkage in G, then we say that x is *infinitely linked to* A *in* G. For $x, y \in V(G)$, and xy*-linkage* of G is a set of pairwise internally disjoint xy-paths of G. If all xy-linkages of G are finite, then x and y are said to be *finitely linked in* G.

A graph is said *block-finite* if all its blocks (i.e. maximal 2-connected subgraphs) are finite. Clearly, G is block-finite if and only if $P_G(x,y)$ is finite for every $x, y \in V(G)$. The *interval* of two vertices x and y of a graph G is the set of vertices of all xy-geodesics (i.e., shortest xy-paths) in G. A graph G is *interval-finite* if all its intervals are finite. For example, it is shown in [1] that any quasi-median graph (see [7] for a definition) is interval-finite. A block-finite graph is obviously interval-finite.

1.2. If G and H are two graphs, a map $f\colon V(G) \to V(H)$ is a *contraction* if f preserves or contracts the edges, i.e., if $f(x) = f(y)$ or $\{f(x), f(y)\} \in E(H)$ whenever $\{x,y\} \in E(G)$. A contraction f from G onto an induced subgraph H of G is a *retraction*, and H is a *retract* of G, if the restriction $f|H$ to H is the identity. A contraction f of G (i.e., from G into itself) *stabilizes* (resp. *strictly stabilizes*) a subgraph H of G, or H is *invariant* (resp. *strictly invariant*) under f, if $f(H) \subseteq H$ (resp. $f(H) = H$). A subset A of $V(G)$ is *invariant* (resp. *strictly invariant*) under a contraction f if the subgraph $G|A$ is.

2. Finite invariant sets of vertices

We begin this section by recalling two results which will be essential in the following.

2.1. Theorem (Schmidt [12]). *Any rayless connected graph G contains a non-empty finite set of vertices which is strictly invariant under any automorphism of G.*

2.2. Theorem (Polat [9, Corollary 2.4]). *Any contraction of a rayless connected graph strictly stabilizes a non-empty finite set of vertices.*

This result will be the cornerstone of the proofs of the subsequent theorems.

2.3. Theorem. *Let $\mathfrak{F}$ be a commuting family of contractions of a connected, interval-finite, rayless graph G. Then there exists a non-empty finite set of vertices which is strictly invariant under any element of $\mathfrak{F}$.*

We need several lemmas.

2.4. We will use the following notation. Let $\mathfrak{F}$ be a family of contractions of a graph G, $f \in \mathfrak{F}$, $x \in V(G)$ and $A \subseteq V(G)$. We will set:

$$[x]_f := \{f^n(x) \colon n \in \mathbb{N}\}$$

$$A_f := \{x \in A : [x]_f \subseteq A \text{ and } f^n(x) = x \text{ for some } n > 0\}$$

$$A_{\mathfrak{F}} := \bigcap_{f \in \mathfrak{F}} A_f.$$

2.5. The usual *distance* in a graph G between two vertices x and y, that is the length of a shortest xy-path in G, will be denoted by $\mathrm{dist}_G(x, y)$. A subgraph H of G is *isometric with* G if $\mathrm{dist}_H(x, y) = \mathrm{dist}_G(x, y)$ for all vertices x and y of H. If G is connected, then obviously so is any isometric subgraph of G.

2.6. Lemma. *Let G be a connected, interval-finite, rayless graph, and A a non-empty subset of $V(G)$ such that $G|A$ is isometric with G. If f is a contraction of G that stabilizes A, then $G|A_f$ is isometric with G.*

Proof. Suppose that A_f is non-empty. Let $x, y \in A_f$, and let P be an xy-geodesic of G included in $G|A$; such a path exists since $G|A$ is isometric with G. For each $n \geqslant 0$ such that $f^n(x) = x$ and $f^n(y) = y$, $f^n(P)$ is an xy-geodesic of $G|A$, since f stabilizes A. Hence, as $G|A$ is interval-finite, there exist $p < q$ such that $f^{pn}(P) = f^{qn}(P) =: Q$, thus $f^{(q-p)n}(Q) = f^{qn}(P) = Q$. Therefore $V(Q) \subseteq A_f$, which proves that $G|A_f$ is an isometric subgraph of G. $\square$

2.7. Let G be a graph. We will endow the vertex set of G with a topology. For $A \subseteq V(G)$ we denote by $\bar{A}$ the set of vertices of G which belong to A or which are infinitely linked to A in G, i.e., by Menger's theorem, $x \in \bar{A}$ if and only if A meets the component of x in the graph $G - S$ for any finite subset S of $V(G - x)$. By [8, 2.4], $A \mapsto \bar{A}$ is the closure operator of a topology on $V(G)$. In the following we will suppose that $V(G)$ is endowed with this topology. Note that $V(G)$ is clearly a T_1-space, and that *$V(G)$ is Hausdorff if and only if the vertices of G are pairwise finitely linked in G.* By [8, 3.2], *$V(G)$ is compact if G is connected and rayless.*

2.8. We will now recall the concept of the Cantor–Bendixson derivative. Let $\mathfrak{A}$ be a topological space. We denote by $\mathfrak{A}'$ the *derivative* of $\mathfrak{A}$, i.e., the set of cluster points of $\mathfrak{A}$. The *Cantor–Bendixson derivative of order* α of $\mathfrak{A}$, $\mathfrak{A}^{(\alpha)}$, is defined by induction as follows:

- $\mathfrak{A}^{(0)} := \mathfrak{A}$
- $\mathfrak{A}^{(\alpha+1)} := (\mathfrak{A}^{(\alpha)})'$
- $\mathfrak{A}^{(\alpha)} := \bigcap_{\beta<\alpha} \mathfrak{A}^{(\beta)}$ if α is a limit ordinal.

In view of the fact that the $\mathfrak{A}^{(\alpha)}$'s form a decreasing sequence, there exists an ordinal α such that $\mathfrak{A}^{(\alpha)} = \mathfrak{A}^{(\alpha+1)}$. The smallest of these ordinals, denoted by $r(\mathfrak{A})$, is the *Cantor–Bendixson rank* of A, and the set $\mathfrak{A}^{(r(A))}$ is the *perfect kernel* of $\mathfrak{A}$.

2.9. Lemma (Polat [8, 3.3 and 3.5]). *If G is a rayless connected graph, then $r(V(G))$ is a successor ordinal with $V(G)^{(r(V(G)))} = \emptyset$.*

2.10. Lemma. *Let G be a rayless connected graph, and let $(C_i)_{i\in I}$ be an infinite family of subsets of $V(G)$ such that, for any finite non-empty $J \subseteq I$, $C_J := \bigcap_{j\in J} C_j$ is a non-empty set such that $G|C_J$ is connected. Then $\bigcap_{i\in I} C_i \neq \emptyset$.*

Proof. Let J be a finite non-empty subset of I. The space C_J is compact since $G|C_J$ is connected and rayless, and, by Lemma 2.9, $r(C_J)$ is a successor ordinal $\alpha_J + 1$, and $C_J^{(\alpha_J+1)} = \emptyset$. Hence $C_J^{(\alpha_J)}$ is a compact set with an empty derivative, thus is finite. Suppose that $\bigcap_{i\in I} C_i = \emptyset$. Then there is a finite J' with $J \subseteq J' \subseteq I$ such that $C_J^{(\alpha_J)} \cap C_{J'} = \emptyset$. Thus $r(C_{J'}) < r(C_J)$, since clearly $C_J^{(\alpha)} \cap C_{J'} \subseteq \bigcap_{\beta\leqslant\alpha} C_{J'}^{(\beta)}$ for every $\alpha \leqslant r(C_J)$. Therefore we can construct inductively an infinite sequence $J_0 \supset J_1 \supset \cdots$ of finite subsets of I such that $r(C_{J_{n+1}}) < r(C_{J_n})$ for every $n \geqslant 0$. So we get an infinite strictly decreasing sequence of ordinals, which is impossible. Thus $\bigcap_{i\in I} C_i \neq \emptyset$. $\square$

2.11. Proof of Theorem 2.3. For every $f \in \mathfrak{F}$, the set A_f, where A stands for $V(G)$, is non-empty by Theorem 2.2, and such that $G|A_f$ is isometric with G by Lemma 2.6. If $g \in \mathfrak{F}$ commutes with f on A_f, and if $x \in A_f$, then $f^p(g(x)) = g(f^p(x)) = g(x)$ for any $p \geqslant 0$ such that $f^p(x) = x$. Thus $g(A_f) \subseteq A_f$; hence, by Theorem 2.2, since $G|A_f$ is rayless and connected, g strictly stabilizes a non-empty finite subset of A_f. Therefore $A_f \cap A_g = (A_f)_g\ (= (A_g)_f)$ is non-empty with $G|(A_f \cap A_g)$ isometric with G. Note that $[x]_f \cup [x]_g \subseteq A_f \cap A_g$ for every $x \in A_f \cap A_g$, hence the restrictions of f and of g to

$A_f \cap A_g$ are automorphisms of $G|(A_f \cap A_g)$. Inductively, for any non-empty finite $\mathfrak{H} := \{f_1, \ldots, f_n\} \subseteq \mathfrak{F}$, the set $A_H := \bigcap_{f \in H} A_f = (\ldots(A_{f_1})\ldots)_{f_n}$ is non-empty and such that $G|A_H$ is isometric with G. Therefore $A_{\mathfrak{F}} \neq \emptyset$ by Lemma 2.10. Furthermore, the restriction of every $f \in \mathfrak{F}$ to $A_{\mathfrak{F}}$ is an automorphism of $G|A_{\mathfrak{F}}$. Since each $G|A_f$ is isometric and G is interval finite, we conclude that $G|A_{\mathfrak{F}}$, being the intersection of all $G|A_f$'s, also is isometric, and hence connected. Consequently, by Theorem 2.1, $A_{\mathfrak{F}}$ contains a non-empty finite subset which is strictly invariant under every element of $\mathfrak{F}$. □

3. Finite invariant simplices

In this section we will consider three particular kinds of graphs, the first and most important ones being *strongly dismantlable graphs*. We recall: (1) that a vertex x is *dominated by a vertex y in a graph G* if y is adjacent with x and with all neighbors of x in G; (2) that a finite graph G is *dismantlable* if its vertices can be linearly ordered $x_0, \ldots, x_n$ so that, for each $i < n$, the vertex x_i is dominated by a vertex $y \neq x_i$ in the subgraph of G induced by $\{x_i, \ldots, x_n\}$. The concept of dismantlability can be straightforwardly extended to infinite graphs as follows:

3.1. Definition. A graph G is said to be *dismantlable* if there is a well-ordering $\leqslant$ on $V(G)$ such that any vertex x which is not the greatest element of $(V(G), \leqslant)$ if such a greatest element exists, is dominated by some vertex $y \neq x$ in the subgraph of G induced by the set $\{z \in V(G): x \leqslant z\}$.

As this extension seems much too general to get interesting results, we introduced in [10] the following restricted concept:

3.2. Definition. A graph G is *strongly dismantlable* (or *subretract-collapsible*) if there is a well-ordering $\leqslant$ on $V(G)$ with a greatest element m such that, for every vertex $x \neq m$, there is a strictly increasing finite sequence $x = x_0 < \cdots < x_n = m$ where, for $0 \leqslant i < n$, the vertex x_i is dominated by x_{i+1} in the subgraph of G induced by the set $\{z \in V(G) : x_i \leqslant z\}$.

Note that *any strongly dismantlable graph is connected and dismantlable.* Furthermore, by [9, Theorem 4.4], *any rayless connected dismantlable graph is strongly dismantlable.* Thus in particular *the finite strongly dismantlable graphs are the dismantlable ones.* We now recall a result which will play the same part as Schmidt's result (2.1) in the proof of the next theorem.

3.3. Theorem [9, Theorem A]. *Let G be a rayless strongly dismantlable graph. Then:*

(i) *any contraction of G strictly stabilizes a non-empty finite simplex;*

(ii) *G contains a non-empty finite simplex which is strictly invariant under every automorphism of G.*

In the statement of next theorem as well as in Theorem 3.7, the term 'block-finite' is only used as a short way to say that the set of paths joining any two vertices of the graph is finite. No use of the block-cutpoint tree will be made. As a matter of fact, the image of a block by a contraction is not necessarily included in a block, so a contraction of a graph does not generally induce a contraction of the block-cutpoint tree of this graph.

3.4. Theorem. *Let G be a rayless, block-finite, strongly dismantlable graph. If $\mathfrak{F}$ is a commuting family of contractions of G, then there exists a non-empty finite simplex which is strictly invariant under all elements of $\mathfrak{F}$.*

The proof of this result, as well as that of Theorem 3.10, will be given in Section 4.

3.5. The second class of graphs that we deal with is that of *ball-Helly graphs.* If x is a vertex of a graph G and r a non-negative integer, the set $B_G(x,r) := \{y \in V(G): \mathrm{dist}_G(x,y) \leqslant r\}$ is the *ball of center x and radius r in G.* A *connected* graph G is called a *ball-Helly graph,* if every finite family of pairwise non-disjoint balls of G has a non-empty intersection. There exists a link between ball-Helly graphs and strongly dismantlable ones, which is given in the following result:

3.6. Proposition [9, Theorem 5.3]. *Any rayless ball-Helly graph is strongly dismantlable.*

From 3.4 and 3.6 we get:

3.7. Theorem. *Let G be a rayless, block-finite, ball-Helly graph. If $\mathfrak{F}$ is a commuting family of contractions of G, then there exists a non-empty finite simplex which is strictly invariant under all elements of $\mathfrak{F}$.*

3.8. The last class of graphs that we consider is that of *chordal graphs*, that is, of graphs that contain no induced cycles of length greater than three. We gave in [11] a characterization of the rayless chordal graphs by means of strongly dismantlable graphs.

3.9. Proposition [11, Theorem 3.5]. *Let G be a connected rayless graph. Then G is chordal if and only if every connected induced subgraph of G is strongly dismantlable.*

The combination of this proposition with Theorem 3.4 gives a result analogous to Theorem 3.7 for block-finite chordal graphs. But, as we will see, we can avoid the restriction of 'being block-finite', to get the following more general result:

3.10. Theorem. *Let $\mathfrak{F}$ be a commuting family of contractions of a connected, rayless, chordal graph G. Then there exists a non-empty finite simplex which is strictly invariant under every element of $\mathfrak{F}$.*

As any tree is chordal, we obtain immediately:

3.11. Corollary. *Let $\mathfrak{F}$ be a commuting family of contractions of a rayless tree T. Then there exists a vertex or an edge of T which is fixed by every element of $\mathfrak{F}$.*

4. Proofs of the results of Section 3

In order to work more easily with rayless strongly dismantlable graphs, we will use another class of graphs that we will recall.

4.1. For an ordinal α we denote by P_α the graph whose vertex set is $V(P_\alpha) = \alpha + 1$ and edge set is $E(P_\alpha) = \{\{\beta, \beta + 1\}: \beta < \alpha\}$.

If G is a graph, a contraction $F: G \times P_\alpha \to G$ will be said to be *continuous* if F is a continuous function from the product space $V(G \times P_\alpha)$ into $V(G)$ when the set $V(G)$ is endowed with the discrete topology, and $\alpha + 1$ with the usual order topology for which $\{(\gamma, \beta]: \gamma < \beta \leqslant \alpha\} \cup \{[0, \beta]: \beta \leqslant \alpha\}$ is a base. That means that F is continuous if and only if, for any $x \in V(G)$ and any limit ordinal $\beta \leqslant \alpha$, there is $\gamma(x) < \beta$ such that $\gamma(x) \leqslant \gamma \leqslant \beta$ implies $F(x, \gamma) = F(x, \beta)$.

4.2. Definition. A graph G is said to be *contractible* if there are an ordinal σ, a vertex a, and a continuous contraction $F: G \times P_\sigma \to G$ such that $F(x, 0) = x$ and $F(x, \sigma) = a$ for every $x \in V(G)$.

4.3. Proposition (Polat [19, Theorem 4.4]). *Let G be a rayless, non-empty, connected graph. The following are equivalent:*

(i) *G is strongly dismantlable;*
(ii) *G is dismantlable;*
(iii) *G is contractible.*

4.4. Proposition. *Any retract of a contractible graph is contractible.*

Proof. Let ρ be a retraction of a contractible graph G onto one of its subgraph H. As G is contractible, there exist an ordinal σ, a vertex a of G, and a continuous contraction $F: G \times P_\sigma \to G$ such that $F(x, 0) = x$ and $F(x, \sigma) = a$ for every $x \in V(G)$. Denote by F' the restriction of $\rho \circ F$ to $V(H \times P_\sigma)$. It is straightforward to check that F' is a continuous contraction from $H \times P_\sigma$ onto H with $F'(x, 0) = x$ and $F'(x, \sigma) = \rho(a)$ for every $x \in V(H)$, which proves that H is contractible. $\square$

Propositions 4.3 and 4.4 imply immediately:

4.5. Corollary. *Any retract of a rayless strongly dismantlable graph is strongly dismantlable.*

4.6. Lemma. *Let G be a connected, block-finite, rayless graph, and A a non-empty subset of $V(G)$ such that $G|A$ is strongly dismantlable. If f is a contraction of G that stabilizes A, then A_f is non-empty and there exists a retraction ρ_f from $G|A$ onto $G|A_f$ (thus $G|A_f$ is strongly dismantlable by* 4.5).

In the following, if H is a subgraph of a graph G, and X a subgraph of $G - H$, the set $\mathfrak{B}(H,X) := \{x \in V(H)\colon V(x;G) \cap V(X) \neq \emptyset\}$ is called the *boundary of H with X*.

Proof. By Theorem 2.1 A_f is non-empty, and by Lemma 2.6 $G|A_f$ is isometric with G, thus is connected. We will assume w.l.o.g. that $A = V(G)$, and we will write G_f for $G|A_f$. Let $H_f := \bigcup_{x,y \in A_f} G(x,y)$. Then H_f is a connected union of blocks of G, and its boundary with any component of $G - H_f$ is a single vertex (in fact, a cut-vertex of G).

(a) *$[x]_f$ is finite for every $x \in V(H_f)$.*

This is obvious if $x \in A_f$. Suppose that $x \in V(H_f) - A_f$. Then x belongs to an ab-path for some $a, b \in A_f$. The result is then a consequence of the facts that $G(a,b)$ is finite and that there are infinitely many integers n such that $f^n(a) = a$ and $f^n(b) = b$, thus such that $f^n(x) \in V(G(a,b))$.

(b) *The boundary of G_f in H_f with any component of $H_f - G_f$ is finite.*

Note that this boundary is the same as the boundary of G_f in G with the corresponding component of $G - G_f$. In the following, for $x \in V(H_f) - A_f$, we will denote by $\mathfrak{C}(x)$ the component of $H_f - G_f$ containing x, and by $B(x)$ the boundary $\mathfrak{B}(G_f, \mathfrak{C}(x))$ of G_f with $\mathfrak{C}(x)$.

Let $x \in V(H_f) - A_f$. By (a), $[x]_f$ is finite, thus $f^n(x) \in A_f$ for some $n > 0$. Let $p := \min\{n\colon f^n(x) \in A_f\}$, and $q := \min\{n > 0\colon f^{p+n}(x) = f^p(x)\}$. By the hypothesis of block-fitniteness, $k := \sup\{|P_G(x, f^{p+n}(x))|\colon 0 \leqslant n \leqslant q\}$ is finite. Suppose that $B(x)$ is infinite. Let $a_0, \ldots, a_k$ be $k+1$ elements of this boundary which do not belong to $[x]_f$. Since $a_i \in A_f$, there exists an integer $m \geqslant p$ such that $f^m(a_i) = a_i$, $i = 0, \ldots, k$. Besides, since $B(x)$ is the boundary of G_f with $\mathfrak{C}(x)$, for every i, there exists an xa_i-path whose only vertex in A_f is a_i. Hence, as G_f is connected, there exist at least $k+1$ $xf^m(x)$-paths in G, contrary to the definition of k.

(c) *Every component of $H_f - G_f$ is finite.*

This is clear since any component X of $H_f - G_f$ is included in $\bigcup_{a,b \in B(x)} G(a,b)$, and since $B(x)$, as well as $G(a,b)$ for every vertices a and b, are finite.

(d) Therefore, by (b) and (c), for every component X of $H_f - G_f$, there exists a least integer $n(X)$ such that $f^{n(X)}(x) \in A_f$ for every $x \in V(X)$, and $f^{n(X)}(b) = b$ for every $b \in \mathfrak{B}(G_f, X)$. Then $f^{n(X)}$ is a retraction from $G|(V(X) \cup \mathfrak{B}(G_f, X))$ onto $G|\mathfrak{B}(G_f, X)$.

Finally, denote by ρ_f the map from $V(G)$ onto A_f defined as follows. Let $x \in A$. If $x \in A_f$, then $\rho_f(x) := x$. If $x \in V(H_f) - A_f$, then $\rho_f(x) := f^{n(\mathfrak{C}(x))}(x)$. If $x \in A - V(H_f)$, and if $y(x)$ is the only element of the boundary of H_f with the component of $G - H_f$ containing x, then $\rho_f(x) := \rho_f(y(x))$. Clearly ρ_f is a retraction from G onto G_f, which completes the proof of the lemma. $\square$

4.7. Proof of Theorem 3.4. We will use the proof of Theorem 2.3 (see 2.11) and will only complete it. For every $\mathfrak{H} \subseteq \mathfrak{F}$, let $G_{\mathfrak{H}} := G|A_{\mathfrak{H}}$. By induction and Lemma 4.6, $G_{\mathfrak{H}}$ is a retract of G for every finite $\mathfrak{H} \subseteq \mathfrak{F}$. Let us show that $G_{\mathfrak{F}}$, which is non-empty (see 2.11), is strongly dismantlable. First note that $G_{\mathfrak{F}}$ is an isometric subgraph of G, thus is connected. W.l.o.g. we can assume that $\mathfrak{F}$ is a semigroup, thus an abelian semigroup. In the following we will use the notation of the proof of Lemma 4.6.

(a) Let $H := \bigcup_{x,y \in A_{\mathfrak{F}}} G(x,y)$, and let $x \in V(H) - A_{\mathfrak{F}}$. Then x belongs to an ab-path for some $a, b \in A_{\mathfrak{F}}$. Since $G(a,b)$ is finite, and $G_{\mathfrak{F}}$ is connected, there exists an element f of $\mathfrak{F}$ such that $G_f(a,b) = G_{\mathfrak{F}}(a,b)$. Thus $\rho_f(x) \in V(G_{\mathfrak{F}}(a,b))$. Now, using the same argument as in part (b) of the proof of Lemma 4.6, we can prove that the boundary of $G_{\mathfrak{F}}$ with the component of $H - A_{\mathfrak{F}}$ containing x is finite. More generally this proves that, for any component X of $G - G_{\mathfrak{F}}$, the boundary $\mathfrak{B}(G_{\mathfrak{F}}, X)$ is finite. Hence, since every $G(a,b)$ is also finite for any pair $\{a,b\}$ of vertices of G, there exists an $f_X \in \mathfrak{F}$ such that $G_{f_X}(a,b) = G_{\mathfrak{F}}(a,b)$ for every $a, b \in \mathfrak{B}(G_{\mathfrak{F}}, X)$, which implies that X is a component of $G - G_{f_X}$.

(b) Now denote by ρ the map from $V(G)$ onto $A_{\mathfrak{F}}$ defined as follows. Let $x \in V(G)$. If $x \in A_{\mathfrak{F}}$, then $\rho(x) := x$. If x belongs to a component X of $G - A_{\mathfrak{F}}$, then $\rho(x) := \rho_{f_X}(x)$. Clearly ρ is a retraction from G onto $G_{\mathfrak{F}}$. Therefore $G_{\mathfrak{F}}$ is strongly dismantlable by Corollary 4.5.

(c) Finally every $f \in \mathfrak{F}$ strictly stabilizes the rayless strongly dismantlable graph $G_{\mathfrak{F}}$, or, in other words, the restriction of each $f \in \mathfrak{F}$ to $V(G_{\mathfrak{F}})$ is an automorphism of $G_{\mathfrak{F}}$. Hence, by Theorem 3.3(ii), $G_{\mathfrak{F}}$, thus G, contains a non-empty finite simplex which is strictly invariant under every element of $\mathfrak{F}$. □

4.8. Lemma. *Any rayless chordal graph is interval-finite.*

Proof. Let G be chordal graph. Suppose that it is not interval-finite, and let a and b be two vertices of G whose interval is infinite and whose distance is minimum with respect to this property. Due to that minimality there exists an infinite ab-linkage $(W_n)_{n \geqslant 0}$ in G where, for every $n \geqslant 0$, $W_n = \langle x_0^n, \ldots, x_d^n \rangle$ is such that $x_0^n = a$, $x_d^n = b$ and $d = \mathrm{dist}_G(a,b)$.

Consider two non-negative integers n and p. Then $W_n \cup W_p$ is a cycle of G. Since G is chordal and since W_n and W_p are ab-geodesics in G, we easily see that the vertices x_i^n and x_i^p must be adjacent for every i, $0 < i < d$. Therefore each infinite set $\{x_i^n : n \geqslant 0\}$ induces an infinite simplex in G, which proves that G is not rayless. □

4.9. Proof of Theorem 3.10. By Lemma 4.8, G is interval-finite, hence, by the proof 2.13 of Theorem 2.3, $G_{\mathfrak{F}} := G|A_{\mathfrak{F}}$ is an isometric subgraph of G, hence it is chordal, rayless and connected. By Proposition 3.9, $G_{\mathfrak{F}}$ is then strongly dismantlable. Consequently, just as in 4.7(c), $G_{\mathfrak{F}}$ contains a non-empty finite simplex which is strictly invariant under every element of $\mathfrak{F}$. □

We will complete this section with a simple result about block-finite graphs, one of the implications occurring already in the proof of Lemma 4.6.

4.10. Proposition. *Let G be a rayless graph. The following are equivalent:*

(i) *G is block-finite;*

(ii) *$P_G(x, y)$ is finite for every $x, y \in V(G)$;*

(iii) *any two vertices x, y of G are finitely linked.*

Proof. (i) and (ii) are obviously equivalent, since, for every $x, y \in V(G)$, $G(x, y)$ is the union of finitely many blocks. The implication (ii) $\Rightarrow$ (iii) is trivial. Assume that (ii) is not true, i.e., there are two vertices a and b such that $G(a, b)$ is infinite. Since $G(a, b)$ is rayless, $V(G(a, b)) - \{a, b\}$ contains an infinite fragmented set X, i.e., there exists a finite subset S of $V(G(a, b))$ which pairwise separates the elements of X. W.l.o.g. we can suppose that $X \cap S = \emptyset$, and that a and b do not belong to a component of $G - S$ containing an element of X. Let $x \in X$. Since x is a vertex of some ab-path, there must be two vertices s and s' in S such that x is a vertex of an ss'-path W of $G(a, b)$ with $V(W) \cap S = \{s, s'\}$. Thus $|S| \geqslant 2$. Therefore, since S is finite and X infinite, there must be two vertices in S, say s_0 and s_1, and infinitely many elements of X, each of them belonging to an s_0s_1-path of $G(a, b)$ having only its endpoints in S. That means that s_0 and s_1 are infinitely linked, thus that (iii) is not true. $\square$

5. Concluding remark

In each of the preceding statements, a 'commuting family' can be clearly replaced by an 'abelian semigroup'. Thus, since any abelian semigroup is left-amenable, it would be interesting to see if those results could be generalized by considering *left-amenable semigroups of contractions* instead of commuting families of contractions, just as Day [12] did with the Markov–Kakutani Theorem.

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